MATH 20D Spring 2023 Lecture 20.

Transforms of Discontinuous Functions and Application to IVP's.

Outline

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- The primary references for midterm 2 is lectures 11-20 and homeworks 4,5, and 6.

- Last time we say how the formulas $\mathscr{L}\{y'(t)\}(s) = s\mathscr{L}\{y(t)\}(s) - y(0)$ and

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  can be applied to give solutions to initial value problems.
- An important feature of $\mathscr{L}$ is that it transforms **differential equations** in t-space into **algebraic equations** in s-space.

Example

Suppose $a \neq 0$, b and c are constants such that $b^2 - 4ac < 0$. Using the method of Laplace transform, solve the initial value problem

$$ay'' + by' + cy = 0, \qquad y(0) = y_0, \quad y'(0) = y_1$$

where y_1 and y_2 are arbitrary constants.

Recall that the *Heaviside step functions* is defined by

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 1. \end{cases}$$

If $f(t)$ is piecewise continuous of exponential order $\alpha > 0$, then

$$\mathscr{L}\{f(t-a)u(t-a)\}(s) = e^{-as}\mathscr{L}\{f(t)\}(s), \qquad s > \alpha$$

where $a \geqslant 0$ is constant.

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(a) $\mathscr{L}\{(t-1)^2u(t-1)\}(s)$,

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(d) $\mathscr{L}\{\Pi_{a,b}(t)\}(s)$ with $0 \leqslant a < b < \infty$ constant and

$$\Pi_{a,b}(t) = \begin{cases} 0, & t < a \text{ or } t > b, \\ 1, & a < t < b. \end{cases}$$

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Using the method of Laplace transform, find an expression for the amount of salt in the tank at time t.

Example

Using the method of Laplace transform, solve the initial value problem

$$y''(t) + 4y(t) = g(t), \qquad y(0) = 0, \quad y'(0) = 0,$$

where

$$g(t) = \begin{cases} 1, & 0 < t < 1, \\ -1, & 1 < t < 2, \\ 0, & 2 < t. \end{cases}$$